In the present case we have two critical cross-sections $\zeta_{c1} = 0.69, \zeta_{c2} = 1.17$ in the range $0 < \zeta < a$, shown in Fig.2 by the dot-dash lines.

The form of the wing surface with the sweep-back angle of $\Lambda = \pi/4$ is shown in Fig.3. A dot denotes the place where parts of the surface couple with the distributed and constant stream functions. Moreover, the figure shows the pressure distribution over the wing surface $p_b(\zeta)$, computed using (2.16) (solid line) and the pressure directly behind the shock wave $p_s(\zeta)$ (the dashed line).

Note that for certain specified configurations of the shock wave, in addition to the sufficient conditions for constructing a smooth wing surface, the equation $\psi_b(\zeta) = \zeta$ also holds at one or several points $\zeta \neq 0$, while additional lines of flow (flow-off) appear in the field of flow together with the line of flow (flow-off) in the plane of symmetry, just as in the case of flow at finite anlges of attack /9, 11/.

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Translated by L.K.

PMM U.S.S.R., Vol.48, No,3, pp. 275-279, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

A VARIATIONAL PRINCIPLE IN THE HYDROMECHANICS OF AN ISOTROPICALLY MAGNETIZABLE MEDIUM*

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The ideas expressed in /1/ are used as the basis for formulating a variational principle for describing the motion of an isotropically magnetizable medium. Representations are obtained for the velocity field, magnetic field and enthalpy written in terms of the Lagrange multipliers. New integrals of the equation of motion are derived.

The system of equations describing the non-relativistic motion of perfect magnetizable media can be written in the form /2/ (*M* is the magnetization of the medium)

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \quad \frac{dS'}{dt} = \frac{d}{dt}(S + S^*) = 0 \tag{1}$$

$$\operatorname{div} \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} - \operatorname{rot}[\mathbf{v}, \mathbf{B}] = 0$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \nabla \psi^{(\rho)} + M \nabla H + \frac{1}{4\pi}[\operatorname{rot} \mathbf{H}, \mathbf{B}]$$

*Prikl.Matem.Mekhan.,48,3,383-387,1984

$$p = p(\rho, S), \quad \mathbf{M} = \frac{1}{4\pi}(\mathbf{B} - \mathbf{H}) = \frac{1}{4\pi}(\mu(\rho, T, H) - 1)\,\mathbf{H}$$

$$\psi^{(\rho)} = -\int_0^H \rho^2 \frac{\partial}{\partial\rho}\left(\frac{M}{\rho}\right) dH = -\frac{H^2}{8\pi} + \frac{1}{4\pi}\int_0^H (\mu - \rho\mu_\rho)\, H\, dH$$

$$S^e = \frac{1}{4\pi\rho}\int_0^H \mu_T H\, dH, \quad \mu_\rho \equiv \frac{\partial\mu}{\partial\rho}, \quad \mu_T \equiv \frac{\partial\mu}{\partial T}$$

Using the relations $TdS = dw - dp/\rho$, we can rewrite the equations of motion of (1) in the form (w is the enthalpy of the system)

$$\frac{\partial\mathbf{v}}{\partial t} - [\mathbf{v}, \operatorname{rot}\mathbf{v}] = -\nabla\left(\frac{v^2}{2} + w + \frac{\psi^{(\rho)} - \psi^{(T)}}{\rho}\right) + T\nabla S' + \frac{1}{4\pi\rho}\left[\operatorname{rot}\frac{\mathbf{B}}{\mu}, \mathbf{B}\right] \tag{2}$$

$$w = U + \frac{p}{\rho}, \quad \psi^{(T)} = -\int_0^H T^2 \frac{\partial}{\partial T}\left(\frac{M}{T}\right) dH$$

We shall show that the equation of motion of the medium can be obtained from the variational principle, provided that the remaining equations of (1) are regarded as constraints imposed on the independent variations $\delta\rho$, δS, $\delta\mathbf{v}$, $\delta\mathbf{B}$. We cannot formulate the variational principle under these conditions without introducing additional constraints, and this can be proved in a manner similar to that used in /3/ for the case of a neutral medium. In the present case the variational principle can be written using Lagrange multipliers $\lambda_1, \ldots, \lambda_4, \lambda_5$ in the form

$$\delta\int_R \Big\{\frac{\rho v^2}{2} - \rho U' + \lambda_1\Big(\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v}\Big) + \lambda_2\Big(\frac{\partial}{\partial t}(\rho S') + \operatorname{div}(\rho S'\mathbf{v})\Big) + \lambda_3\Big(\frac{\partial}{\partial t}(\rho\alpha) + \operatorname{div}(\rho\alpha\mathbf{v})\Big) + \lambda_4 \operatorname{div}\mathbf{B} + \lambda_5\cdot\Big(\frac{\partial\mathbf{B}}{\partial t} - \operatorname{rot}[\mathbf{v}, \mathbf{B}]\Big)\Big\}\, dR = 0 \tag{3}$$

$$U' = U(\rho, S) + U^e, \quad U^e = \frac{BH}{4\pi\rho} - \frac{H^2}{8\pi\rho} - \frac{\psi^{(T)}}{\rho}$$

Here $\alpha = \alpha(\mathbf{r}, t)$ is an arbitrary Lagrangian coordinate of the system, so that $d\alpha/dt = 0$, U' is the total internal energy and U^e is the internal energy of the system determined by the magnetic field.

We assume in the variational principle (3) that the independent variations $\delta\rho$, δS, $\delta\mathbf{v}$, $\delta\mathbf{B}$ vanish at the boundary of the four-dimensional volume R ($dR \equiv dx^3\, dt$).

Computations show that any functions of position and time can be used as the Lagrangian multipliers $\lambda_1, \lambda_4, \lambda_5$ in (3), and specified constraints are imposed on λ_2 and λ_3. The constraints ensure that the equation of motion (2) are satisfied by the expressions for $\mathbf{v}$, $\mathbf{B}$ and the thermodynamic quantities (or for p when the medium is incompressible) given in terms of the Lagrange multipliers, and the expressions follow directly from the variational principle (3).

Indeed, varying (3) and equating to zero the coefficients of the independent variations $\delta\lambda_1, \ldots, \delta\lambda_4, \delta\lambda_5$, we obtain (by virtue of $d\alpha/dt = 0$) the first four equations of system (1). Equating to zero the coefficients of the variations $\delta\mathbf{v}$, $\delta\mathbf{B}$, δS and $\delta\rho$ we obtain, one after the other,

$$\mathbf{v} = \nabla\lambda_1 + S'\nabla\lambda_2 + \alpha\nabla\lambda_3 + \frac{1}{\rho}[\mathbf{B}, \operatorname{rot}\lambda_5] \tag{4}$$

$$\frac{\mathbf{H}}{4\pi} = -\frac{\partial\lambda_5}{\partial t} + [\mathbf{v}, \operatorname{rot}\lambda_5] - \nabla\lambda_4 \tag{5}$$

$$(1 + S_S{}^e)\Big(T + \frac{d\lambda_2}{dt}\Big) = 0 \tag{6}$$

$$w = \frac{v^2}{2} - \frac{d\lambda_1}{dt} - S'\frac{d\lambda_2}{dt} - \alpha\frac{d\lambda_3}{dt} - \frac{\psi^{(\rho)} - \psi^{(T)}}{\rho} \tag{7}$$

The last expression can be written, by virtue of (4), in the form

$$w = -\frac{\partial\lambda_1}{\partial t} - S'\frac{\partial\lambda_2}{\partial t} - \alpha\frac{\partial\lambda_3}{\partial t} - \frac{\psi^{(\rho)} - \psi^{(T)}}{\rho} - \frac{v^2}{2} + \frac{1}{\rho}\mathbf{v}\cdot[\mathbf{B}, \operatorname{rot}\lambda_5] \tag{8}$$

Since generally speaking

$$S_S{}^e = \frac{\partial}{\partial S}\Big(\frac{1}{4\pi\rho}\int_0^H \mu_T H\, dH\Big) \neq -1$$

therefore it follows from (6) that the following constraint must be imposed on λ_2 :

$$d\lambda_2/dt = -T \tag{9}$$

Now we can show that if we also assume that

$$d\lambda_3/dt = 0 \tag{10}$$

then the equation of motion (2) will be satisfied provided that $\mathbf{v}$, $\mathbf{H}$ and w have the representations (4), (5) and (7).

Indeed, using (4) we can write (2) in the form

$$\frac{\partial \mathbf{v}_H}{\partial t} - [\mathbf{v}, \operatorname{rot} \mathbf{v}] = -\nabla\left(\frac{v^2}{2} + w + \frac{\psi^{(\rho)} - \psi^{(T)}}{\rho} + \frac{\partial\lambda_1}{\partial t} + S'\frac{\partial\lambda_2}{\partial t} + \alpha\frac{\partial\lambda_3}{\partial t}\right) + \frac{1}{4\pi\rho}\left[\operatorname{rot}\frac{\mathbf{B}}{\mu}, \mathbf{B}\right] + \left(T + \frac{d\lambda_2}{dt}\right)\nabla S' - \frac{dS'}{dt}\nabla\lambda_2 + \frac{d\lambda_3}{dt}\nabla\alpha - \frac{d\alpha}{dt}\nabla\lambda_3 \tag{11}$$

$$\left(\mathbf{v}_H \equiv \frac{1}{\rho}[\mathbf{B}, \operatorname{rot} \boldsymbol{\lambda}_5]\right)$$

By virtue of the equations $dS'/dt = 0$, $d\alpha/dt = 0$ and (9), (10) and (7), the last expression yields

$$\frac{\partial \mathbf{v}_H}{\partial t} - [\mathbf{v}, \operatorname{rot} \mathbf{v}_H] - \frac{1}{4\pi\rho}\left[\operatorname{rot}\frac{\mathbf{B}}{\mu}, \mathbf{B}\right] + \nabla(\mathbf{v}\cdot\mathbf{v}_H) = 0$$

or

$$\frac{\partial}{\partial t}[\mathbf{b}, \mathbf{a}] - [\mathbf{v}, \operatorname{rot}[\mathbf{b}, \mathbf{a}]] - \left[\operatorname{rot}\frac{\rho\mathbf{b}}{\mu}, \mathbf{b}\right] + \nabla(\mathbf{v}\cdot[\mathbf{b}, \mathbf{a}]) = 0 \tag{12}$$

$$(\operatorname{rot} \boldsymbol{\lambda}_5 \equiv \mathbf{a},\ \mathbf{B}/(4\pi\rho) \equiv \mathbf{b})$$

From (5) we obtain

$$\operatorname{rot}\frac{\rho\mathbf{b}}{\mu} = -\frac{\partial \mathbf{a}}{\partial t} + \operatorname{rot}[\mathbf{v}, \mathbf{a}]$$

and by virtue of the induction equation and the equation $\operatorname{div}\mathbf{B} = 0$, we have

$$\frac{\partial \mathbf{b}}{\partial t} = \operatorname{rot}[\mathbf{v}, \mathbf{b}] + \mathbf{b}\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{b}$$

Then (12) can be written in the form

$$[\mathbf{a}, \operatorname{rot}[\mathbf{b}, \mathbf{v}]] + [\mathbf{b}, \operatorname{rot}[\mathbf{v}, \mathbf{a}]] + [\mathbf{v}, \operatorname{rot}[\mathbf{a}, \mathbf{b}]] = [\mathbf{a}, \mathbf{b}]\operatorname{div}\mathbf{v} + [\mathbf{v}, \mathbf{a}]\operatorname{div}\mathbf{b} + \nabla([\mathbf{a}, \mathbf{b}]\cdot\mathbf{v}) \tag{13}$$

Using the formulas of vector analysis we can show that (13) is an identity.

Thus the equation of motion (2) is satisfied by the representations (4), (5) and (7) for any λ_1, λ_4, λ_5 and λ_2, λ_3 satisfying (9) and (10). This implies that under the constraints formulated above and imposed on the independent variations $\delta\rho$, δS, $\delta\mathbf{v}$, $\delta\mathbf{B}$ and Lagrange multipliers λ_2 and λ_3, the variational principle (3) is equivalent to the system of equations of motion (1).

Bearing in mind the practical applications of the variational principle, we can formulate (3) in a different manner, with the Lagrangian acquiring an interesting physical interpretation.

We shall assume that the variations of the Lagrangian multipliers also vanish at the boundary of the volume R. Then, integrating the expressions in (3) by parts, we obtain the Lagrangian in the form

$$L = \frac{\rho v^2}{2} - \rho U' - \rho\frac{d\lambda_1}{dt} - \rho S'\frac{d\lambda_2}{dt} - \rho\alpha\frac{d\lambda_3}{dt} - \mathbf{B}\cdot\nabla\lambda_4 - \mathbf{B}\cdot\left(\frac{\partial\boldsymbol{\lambda}_5}{\partial t} - [\mathbf{v}, \operatorname{rot}\boldsymbol{\lambda}_5]\right)$$

Taking (5), (7), (9) into account and the expression for $U' = U + U^e$, we obtain

$$L = p + \frac{H^2}{8\pi} + \psi^{(\rho)} = p + p^e = p'$$

where p^e denotes part of the pressure depending exclusively on the magnetic field.

Thus the variational principle can be written in the form

$$\delta\int_R p'(w, S, H)\,dR = 0 \qquad (p' = p(w, S) + p^e) \tag{14}$$

where the total pressure p' in the medium must be regarded as a function of w, S and H, and the representations (7), (6) and (5) with the constraints (9) and (10) must be used in carrying out the variation.

When $\mu = \text{const}$, we have $L = p + B^2/(8\pi)$ and without the field the Lagrangian simply becomes equal to the pressure, as shown in /1/ for the case of standard hydromechanics.

The form (14) represents a simple physical interpretation of the variational principle. Obviously, it is simpler to take ρ, S and H as the basic variables. Then, using expression (8), we can write (14) in the form

$$\delta\int_R \Big\{\Big(\frac{\partial\lambda_1}{\partial t}+S'\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}\Big)\rho+\rho U(\rho,S)- \tag{15}$$

$$\frac{1}{4\pi}\int_0^H(\mu-\mu_T T)H\,dH+\frac{\rho v^2}{2}-\mu\mathbf{v}\cdot[\operatorname{rot}\boldsymbol{\lambda}_5,\mathbf{H}]\Big\}dR=0$$

$$\mathbf{v}=\nabla\lambda_1+S'\nabla\lambda_2+\alpha\nabla\lambda_3+\frac{\mu}{\rho}[\operatorname{rot}\boldsymbol{\lambda}_5,\mathbf{H}]$$

$$\mu=\mu(\rho,T(\rho,S),H),\qquad S'=S+\frac{1}{4\pi\rho}\int_0^H\mu_T H\,dH$$

$$\frac{d\lambda_3}{dt}=\frac{d\alpha}{dt}=0,\qquad \frac{d\lambda_2}{dt}=-T(\rho,S)$$

which is more suitable for computations.

In the case of an incompressible fluid ($d\rho/dt=0$, $\operatorname{div}\mathbf{v}=0$) we must omit from (1) the equations of state $p=p(\rho,S)$ and entropy S, and replace the enthalpy w by p/ρ.

Thus, the variational principle in the form (15) becomes

$$\delta\int_R\Big\{\rho\Big(\frac{\partial\lambda_1}{\partial t}+S^e\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}\Big)+\frac{\rho v^2}{2}+\mu\mathbf{v}\cdot[\mathbf{H},\operatorname{rot}\boldsymbol{\lambda}_5]- \tag{16}$$

$$\frac{1}{4\pi}\int_0^H(\mu-T\mu_T)H\,dH\Big\}dR=0$$

$$\mathbf{v}=\nabla\lambda_1+S^e\nabla\lambda_2+\alpha\nabla\lambda_3-\frac{\mu}{\rho}[\mathbf{H},\operatorname{rot}\boldsymbol{\lambda}_5],\qquad \mu=\mu(\rho,T,H)$$

$$S^e=\frac{1}{4\pi\rho}\int_0^H\mu_T H\,dH,\qquad \frac{d\alpha}{dt}=\frac{d\lambda_3}{dt}=0,\qquad \frac{d\lambda_2}{dt}=-T,\qquad \delta\rho=0$$

for the incompressible fluid. When $\mu=1$, (15) yields the following variational principle for the magnetohydrodynamics of a non-magnetizable compressible medium:

$$\delta\int_R\Big\{\rho\Big(\frac{\partial\lambda_1}{\partial t}+S\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}\Big)+\rho U(\rho,S)+\frac{\rho v^2}{2}-\frac{H^2}{8\pi}-\mathbf{v}\cdot[\operatorname{rot}\boldsymbol{\lambda}_5,\mathbf{H}]\Big\}dR=0$$

and in the case of an incompressible fluid ($\delta\rho=0$) we have

$$\delta\int_R\Big\{\rho\Big(\frac{\partial\lambda_1}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}\Big)+\frac{\rho v^2}{2}-\frac{H^2}{8\pi}-\mathbf{v}\cdot[\operatorname{rot}\boldsymbol{\lambda}_5,\mathbf{H}]\Big\}dR=0.$$

When $H=0$, the formulations (3) and (16) yield the expresions given in /1/, and for $\mu=1$, (3) yields an expression obtained in /4/ where the additional assumption that $\alpha=0$ was made.

Following /5/ we note, that the equations of motion of a perfect magnetizable medium can be written in the form resembling the Hamiltonian form of the equations of motion of a system of material points.

By virtue of the above arguments, expression (12) represents an identity, and, therefore, the equation of motion (11) can be written in the form

$$\nabla\Big(\frac{v^2}{2}+w+\frac{\psi^{(\rho)}-\psi^{(T)}}{\rho}+\frac{\partial\lambda_1}{\partial t}+S'\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}-\mathbf{v}\cdot\mathbf{v}_H\Big)= \tag{17}$$

$$\Big(T+\frac{d\lambda_2}{dt}\Big)\nabla S'+\frac{d\lambda_3}{dt}\nabla\alpha-\frac{d\alpha}{dt}\nabla\lambda_3$$

The above expression presupposes only the representation

$$\mathbf{v}=\nabla\lambda_1+S\nabla\lambda_2+\alpha\nabla\lambda_3+\mathbf{v}_H,\quad \mathbf{v}_H=\frac{1}{\rho}[\mathbf{B},\operatorname{rot}\boldsymbol{\lambda}_5'] \tag{18}$$

Let us now assume that $d\lambda_2/dt=-T$. Then, writing

$$h\equiv\frac{v^2}{2}+w+\frac{\psi^{(\rho)}-\psi^{(T)}}{\rho}+\frac{\partial\lambda_1}{\partial t}+S'\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}-\mathbf{v}\cdot\mathbf{v}_H$$

we obtain from (17)

$$h=h(t,\alpha,\lambda_3),\qquad \frac{\partial h}{\partial\alpha}=\frac{d\lambda_3}{dt},\qquad \frac{\partial h}{\partial\lambda_3}=-\frac{d\alpha}{dt}$$

If we now write $\mathbf{v}$ in the form (18) where $\lambda_3(\mathbf{r},t)$, $\alpha(\mathbf{r},t)$, $\lambda_2(\mathbf{r},t)$ are such that

$$\frac{d\alpha}{dt}=\frac{d\lambda_3}{dt}=0, \quad \frac{d\lambda_2}{dt}=-T$$

then from (17) we obtain the integral of the equation of motion

$$h=f(t) \tag{19}$$

In the case of standard gas dynamics the integral becomes

$$\frac{v^2}{2}+w+\frac{\partial\lambda_1}{\partial t}+S\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}=f(t) \tag{20}$$

while in the case of magnetohydrodynamics $(\mu=1)$ we have

$$\frac{v^2}{2}+w+\frac{\partial\lambda_1}{\partial t}+S\frac{\partial\lambda_2}{\partial t}+\alpha\frac{\partial\lambda_3}{\partial t}-\frac{1}{\rho}\mathbf{v}\cdot[\mathbf{H},\operatorname{rot}\lambda_5]=f(t) \tag{21}$$

We note that the integrals (19)–(21) also exist for the vortical flow $(\operatorname{rot}\mathbf{v}\neq 0)$, where the orientation of the field $\mathbf{H}$ relative to the stream lines is arbitrary. The determination of $\lambda_1,\ldots,\lambda_5$ in (4) and (5) is essential. It is clear that the form of these potentials is not defined uniquely.

In the case of an irrotational motion $(\mathbf{v}=\nabla\lambda_1)$ (20) yields the usual Lagrange integral. Finally, we note that in the case of stationary motion along the stream lines a generalized Bernoulli integral

$$\frac{v^2}{2}+w+\frac{\psi^{(\rho)}-\psi^{(T)}}{\rho}-\frac{\mu}{\rho}\mathbf{v}\cdot[\mathbf{H},\operatorname{rot}\lambda_5]=\text{const}$$

exists, where the constant depends, in general, on the stream line and the orientation of the field $\mathbf{H}$ is arbitrary. This implies that when $\mathbf{v}\parallel\mathbf{H}$, then the last term on the left-hand side vanishes, while when $\mathbf{v}\perp\mathbf{H}$, the term becomes $BH/(4\pi\rho)$ by virtue of (5) (in complete agreement with the results known in magnetohydrodynamics).

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Translated by L.K.